SOLUTION EXERCISE SHEET 23

Exercise 1. In this exercise we aim to show that for $f \in C^2(\mathbb{R}^3;\mathbb{R}^3)$ the following identities hold true

$$\nabla\cdot(\nabla\times f) = \operatorname{div}(\operatorname{rot} f) = 0, \quad \nabla\times(\nabla\times f) = \operatorname{rot}(\operatorname{rot} f) = \nabla(\nabla\cdot f) - \Delta f = \operatorname{grad}\operatorname{div} f - \Delta f.$$

This is achieved through direct computation. For the first one we have that

$$\operatorname{rot} f = \begin{pmatrix} \partial_{x_2} f_3 - \partial_{x_3} f_2 \\ -\partial_{x_1} f_3 + \partial_{x_3} f_1 \\ \partial_{x_1} f_2 - \partial_{x_2} f_1 \end{pmatrix}$$

$$\implies \operatorname{div}(\operatorname{rot} f) = \partial_{x_1}(\partial_{x_2} f_3 - \partial_{x_3} f_2) + \partial_{x_2}(-\partial_{x_1} f_3 + \partial_{x_3} f_1) + \partial_{x_3}(\partial_{x_1} f_2 - \partial_{x_2} f_1) = 0.$$

Next, we have that

$$\begin{aligned}
\operatorname{rot}(\operatorname{rot} f) &= \begin{pmatrix} \partial_{x_2}(\operatorname{rot} f)_3 - \partial_{x_3}(\operatorname{rot} f)_2 \\ -\partial_{x_1}(\operatorname{rot} f)_3 + \partial_{x_3}(\operatorname{rot} f)_1 \\ \partial_{x_1}(\operatorname{rot} f)_2 - \partial_{x_2}(\operatorname{rot} f)_1 \end{pmatrix} \\
&= \begin{pmatrix} \partial_{x_2}(\partial_{x_1} f_2 - \partial_{x_2} f_1) - \partial_{x_3}(-\partial_{x_1} f_3 + \partial_{x_3} f_1) \\ -\partial_{x_1}(\partial_{x_1} f_2 - \partial_{x_2} f_1) + \partial_{x_3}(\partial_{x_2} f_3 - \partial_{x_3} f_2) \\ \partial_{x_1}(-\partial_{x_1} f_3 + \partial_{x_3} f_1) - \partial_{x_2}(\partial_{x_2} f_3 - \partial_{x_3} f_2) \end{pmatrix} \\
&= \begin{pmatrix} \partial_{x_1}(\partial_{x_2} f_2 + \partial_{x_3} f_3) - (\partial_{x_2x_2} f_1 + \partial_{x_3x_3} f_1) \\ \partial_{x_2}(\partial_{x_1} f_1 + \partial_{x_3} f_3) - (\partial_{x_1x_1} f_2 + \partial_{x_3x_3} f_2) \\ \partial_{x_3}(\partial_{x_1} f_1 + \partial_{x_2} f_2) - (\partial_{x_1x_1} f_3 + \partial_{x_2x_2} f_3) \end{pmatrix} \\
&= \nabla(\operatorname{div} f) - \Delta f.
\end{aligned}$$

This proves the two identities.

Exercise 2. In this exercise the important observation is that we can simplify the expression of the rot by taking one component of g to be zero and that f being divergence free implies that we can express $\partial_{x3} f_3$ with $\partial_{x1} f_1 + \partial_{x2} f_2$. Define the map $g : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$g(x) = \begin{pmatrix} \int_0^{x_3} f_2(x_1, x_2, y)\,dy \\ -\int_0^{x_3} f_1(x_1, x_2, y)\,dy \\ 0 \end{pmatrix}.$$

This map is clearly $C^1(\mathbb{R}^3;\mathbb{R}^3)$ as f is $C^1(\mathbb{R}^3;\mathbb{R}^3)$ and $\operatorname{rot} g = f$ as

$$\operatorname{rot} g(x) = \begin{pmatrix} -\partial_{x_3} g_2(x) \\ \partial_{x_3} g_1(x) \\ \partial_{x_1} g_2(x) - \partial_{x_2} g_1(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \\ -\int_0^{x_3} (\partial_{x_1} f_2 + \partial_{x_2} f_1)(x_1, x_2, y) dy \end{pmatrix}$$
$$= \begin{pmatrix} f_1(x) \\ f_2(x) \\ \int_0^{x_3} \partial_{x_3} f_3(x_1, x_2, y) dy \end{pmatrix} = f - f_3(x_1, x_2, 0) e_3.$$

This map is almost a vector potential for f. There is only one additional term in the last component. There are many ways to solve this problem as a vector potential is only unique modulo scalar potential. One possibility is by adding $\int_0^{x_1} f_3(y, x_2, 0) dy$ to the second component of g. In that case g has clearly still the same regularity than before and the computation from above is still valid with an extra term $\partial_{x_1} \int_0^{x_1} f_3(y, x_2, 0) dy = f_3(x_1, x_2, 0)$ in the last term.

Exercise 3. Consider the map $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{|x|}$. This map is clearly $C^\infty(\mathbb{R}^3 \setminus \{0\})$. By straight forward computations we have that the gradient of $x \in \mathbb{R}^3 \setminus \{0\} \mapsto |x|$ is given by $\frac{x}{|x|}$. Thus we have by the chain rule that

$$\nabla f(x) = -\frac{x}{|x|^3}, \quad \forall x \in \mathbb{R}^3 \setminus \{0\}$$
$$\partial_{x_i x_i} f = -\frac{1}{|x|^3} - x_i \cdot \frac{-3x_i}{|x|^5} = \frac{-|x|^2 + 3x_i^2}{|x|^5}, \quad \forall x \in \mathbb{R}^3 \setminus \{0\}, i = 1, 2, 3.$$

Thus we get that

$$\Delta f(x) = \sum_{i=1}^3 \frac{-|x|^2 + 3x_i^2}{|x|^5} = \frac{-3|x|^2 + 3\sum_{i=1}^3 x_i^2}{|x|^5} = 0.$$

Exercise 4. There are two different ways one can solve this exercise. One possibility is by taking an open cover $(V_x)_{x\in S}$ such that on each $V_x \cap S$ we have a regular parametrization. Then we consider a second open cover $(U_x)_{x\in S}$ such that $\overline{U_x} \subset V_x$, this is possible by reducing the size of well chosen balls. By compactness of S we can reduce this cover to a finite subcover $(U_i)_{i=1,\dots,n}$. Then we can take a partition of the unity $(\eta_i)_{i=1,\dots,n}$ for U_i, V_i by proposition 8.16 (it is not difficult to check that the U_i can be taken such that all assumption of proposition 8.16 hold true). Further, observe that the partition constructed proposition 8.16 is such that $\sum_{i=1}^n \eta_i = 1$ on $\bigcup_{i=1}^n U_i$. As this set is open we get that $\operatorname{rot}(f) = \operatorname{rot}(\sum_{i=1}^n f\eta_i)$ on $U := \bigcup_{i=1}^n U_i$. As $S \subset U$ we get that

$$\int_S \operatorname{rot}(f) \cdot \mathbf{n} d\sigma = \sum_{i=1}^n \int_S \operatorname{rot}(f\eta_i) \cdot \mathbf{n} d\sigma = \sum_{i=1}^n \int_{S\cap V_i} \operatorname{rot}(f\eta_i) \cdot \mathbf{n} d\sigma.$$

Finally, as on $S \cap V_i$ we have a parametrization, we can apply Kelvin-Stokes to conclude that

$$\int_{S \cap V_i} \operatorname{rot}(f\eta_i) \cdot \mathbf{n} d\sigma = \int_{\partial(S \cap V_i)} f\eta_i \cdot ds = 0$$

as $\eta_i \in C_c^\infty(V_i)$.

The other possibility to prove the statement is by considering a regularization of f and the divergence theorem. We do not present the details for that approach but a sketch goes as follow. First, one shows integration by parts in $\mathbb{R}^n$ through the divergence theorem, i.e. if $\Omega \subseteq \mathbb{R}^n$ has C^1-boundary and $u, v \in C^1(\overline{\Omega})$, then

$$\int_\Omega u v_{x_i} dx = \int_{\partial\Omega} uv\, n_i ds - \int_\Omega v u_{x_i} dx.$$

Using integration by parts we can show that in case of $f \in C^1(\mathbb{R}^3; \mathbb{R}^3)$, we have that the family of ϵ-mollification is such that $f_\epsilon \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$, $\partial_{x_i}(f_\epsilon) = (\partial_{x_i} f)_\epsilon$ and $f_\epsilon, \partial_{x_i} f_\epsilon \xrightarrow{\epsilon \to 0} f, \partial_{x_i} f$ locally uniformly on $\mathbb{R}^3$ for every $i = 1, 2, 3$. Then we get that on the compact set S it holds that

$$\int_S \operatorname{rot}(f_\epsilon) \cdot \mathbf{n} d\sigma \xrightarrow{\epsilon \to 0} \int_S \operatorname{rot} f \cdot \mathbf{n} d\sigma.$$

But, by the divergence theorem we have that

$$\int_S \operatorname{rot}(f_\epsilon) \cdot \mathbf{n} d\sigma = \int_\Omega \operatorname{div} \operatorname{rot}(f_\epsilon) dx = 0,$$

where $\Omega \subseteq \mathbb{R}^3$ is the interior of S, i.e $S = \partial\Omega$.